Abstract

This paper presents a new computer method for the rapid design and ultimate strength capacity evaluation of composite steel-concrete cross-sections with arbitrary shape. The proposed approach has been found to be very stable for all cases examined herein even when the section is close to the state of pure compression or tension, and it is not sensitive to the initial or starting values, to how the origin of the reference loading axes is chosen and to the strain softening exhibited by concrete in compression. A computer program was developed, aimed at obtaining the ultimate strength and reinforcement required by composite cross-sections subjected to combined biaxial bending and axial load. In order to illustrate the accuracy and efficiency of the proposed method, this program was used to study several representative examples, which have been studied previously by other researchers. The examples run and the comparisons made prove the effectiveness and time saving of the proposed method of analysis.

Keywords: interaction diagrams, rapid design, composite cross-sections, ultimate strength analysis, arc-length method, fully yield surfaces, biaxial bending.

1 Introduction

In recent years, some methods have been presented for the ultimate strength analysis and design of various concrete and composite steel-concrete sections [1-7]. Among several existing techniques that determine the biaxial interaction diagrams, two are the most common. The first consists of a direct generation of points of the failure surface by varying the position and inclination of the neutral axis and imposing a strain distribution corresponding to a failure condition. This technique generates the failure surface through 3D curves [4], making the application of this technique rather cumbersome for practical applications. The second approach is based upon the
solution of the non-linear equilibrium equations according to the classical Newton’s scheme consisting of an iterative sequence of linear predictions and non-linear corrections to obtain either the strain distribution or the location and inclination of the neutral axis which determines the ultimate load, where one or two components of the section forces can remain constant. In general, these methods generate plane interaction curves and give fast solutions but are sensitive to the origin of the loading axes and some problems in convergence may arise, particularly when the initial or starting values of the variables are not selected properly and under large axial forces [3].

Numerical procedures that allow the design of steel reinforcements for sections subjected to biaxial bending and axial force have been proposed in several papers [1-3]. Dundar and Sahin [1] presented a procedure based on the Newton-Raphson method for dimensioning of arbitrarily shaped reinforced concrete sections, subjected to combined biaxial bending and axial compression. In their analysis the distribution of concrete stress is assumed to be rectangular according with Whitney’s rectangular block. Rodrigues and Ochoa [2] extended this method for the more general cases when the concrete in compression is modelled using the explicit nonlinear stress-strain relationships. Chen et.al. [3] proposed an iterative quasi-Newton procedure based on the Regula-Falsi numerical scheme for the rapid sectional design of short concrete-encased composite columns of arbitrary cross-section subjected to biaxial bending. This algorithm is limited to fully confined concrete. For all these algorithms, problems of convergence may arise especially when starting or initial values (i.e. the parameters that define the strain profile over the cross-section) are not properly selected, and they may become unstable near the state of pure compression or tension. The main objective of this paper is to present a new formulation by which the biaxial interaction diagrams and moment capacity contours of a composite steel-concrete cross-section can be determined, which make use of an incremental-iterative procedure based on arc-length constraint equations. Furthermore, a new procedure based on Newton iterative method is proposed to design the required reinforcement for sections subjected to axial force and biaxial bending moments.

Considering the solution of non-linear equilibrium equations, involved in the evaluation of the interaction diagrams, the proposed incremental-iterative method is advantageous with respect to the existing ones [2, 3], in that the solution is obtained by solving just two coupled nonlinear equations and the convergence stability is not sensitive to the initial/starting values of the basic variables (i.e. the ultimate curvatures about global axes $\phi_x$ and $\phi_y$), involved in the iterative process. For all cases examined herein has been found that starting the iterative process with the initial curvatures $\phi_x=0$ and $\phi_y=0$ the convergence is achieved in a low number of iterations. Most of the existing methods that use iterative procedures [2, 3] are not general, they can compute either interaction curves for a given bending moment ratio [2] or load contours for a given axial load [3]. As it will be presented in the next sections, the proposed approach can handle both types of interaction curves.

Moreover using the proposed method we have found that near the axial load capacity under pure compression, when the strain-softening of the concrete is taken into account, the solution is not unique which implies non-convexity of the failure
surface in these situations. Therefore, the proposed approach based on arc-length constraint strategy is essential to assure the convergence of the entire process and to determine all possible solutions.

The method proposed herein, for design of cross-sections, consists of computing the required reinforcement area \( A_{\text{tot}} \) supposing that all structural parameters are specified and, under given external loads, the cross section reaches its failure either in tension or compression. The problem is formulated by means of three equilibrium equations for the section. The condition of ultimate limit state is enforced by a compatibility equation imposing the maximum strain on the section to be equal to the limit strain of the corresponding material. These nonlinear equations are manipulated so that one of them is uncoupled and the Newton iterative strategy is applied only to the remaining coupled equilibrium equations. The proposed approach is advantageous with respect to the existing ones [1,2,3], in that the solution is obtained by solving just three coupled nonlinear equations [3] and the convergence stability is not sensitive to the initial/starting values of the basic variables [1,2,3] (i.e. the ultimate sectional curvatures \( \phi_x \) and \( \phi_y \)) involved in the iterative process. This method, as compared to other iterative methods used in the solution of nonlinear equations for design of cross-sections, is very stable, converging rapidly if the initial value of the required reinforcement area, \( A_{\text{tot}} \), is properly selected. The stability and rapid convergence of the proposed approach is also due to the fact that the Jacobian’s of the nonlinear system of equations is explicitly computed whereas in the methods proposed by Dundar and Sahin [1] and Rodrigues and Ochoa [2], the partial derivatives, involved in determination of the Jacobian matrix, can be only approximately expressed in terms of finite differences.

2 Mathematical formulation

2.1 Basic assumptions

Consider the cross-section subjected to the action of the external bending moments about both global axes and axial force as shown in Figure 1. The cross-section may assume any shape with multiple polygonal or circular openings. It is assumed that plane section remains plane after deformation. This implies perfect bond between the steel and concrete components of a composite concrete-steel cross section. Thus resultant strain distribution corresponding to the curvatures about global axes \( \Phi=\{\phi_x, \phi_y\} \) and the axial compressive strain \( \varepsilon_0 \) can be expressed at a generic point \((x, y)\) in a linear form as:

\[
\varepsilon = \varepsilon_0 + \phi_x y + \phi_y x
\]  

(1)
The constitutive relations for concrete under compression are represented by a combination of a second-degree parabola and a straight line as depicted in Fig. 2a. The parameter \( \gamma \) represents the degree of confinement in the concrete, and allows for the modelling of creep and confinement in the concrete by simply varying the crushing strain \( \varepsilon_{c,0} \) and \( \gamma \) respectively. The tensile strength of concrete is neglected. A multi-linear elasto-plastic stress-strain relationship, both in tension and in compression, is assumed for the structural steel and the steel reinforcing bars (Fig. 2b).

At ultimate strength capacity the equilibrium is satisfied when the external forces are equal to the internal ones and in the most compressed or tensioned point the ultimate strain is attained. These conditions can be represented mathematically in terms of the following nonlinear system of equations as:
\[
\begin{align*}
\int_\mathcal{A} \sigma(x, y, \phi_x, \phi_y) \, dA - N &= 0 \\
\int_\mathcal{A} \sigma(x, y, \phi_x, \phi_y) \, x \, dA - M_x &= 0 \\
\int_\mathcal{A} \sigma(x, y, \phi_x, \phi_y) \, y \, dA - M_y &= 0 \\
\varepsilon_0 + \phi_x(y(x, \phi_y) + \phi_y(x(x, \phi_x)) - \varepsilon_u &= 0
\end{align*}
\] (2)

and in which \(N, M_x, M_y, \varepsilon_0, \phi_x, \phi_y\) represents the unknowns. In the Eqs. (2) the first three relations represent the basic equations of equilibrium for the axial load \(N\) and the biaxial bending moments \(M_x, M_y\) respectively, given in terms of the stress resultants. The last equation represents the ultimate strength capacity condition; that is, in the most compressed or most tensioned point the ultimate strain is attained, and in which \(x(x, \phi_y)\) and \(y(y, \phi_x)\) represents the coordinates of the point in which this condition is imposed.

### 2.2 Determination of the interaction diagrams and moment capacity contours

Under the above assumptions, the problem of the ultimate strength analysis of composite cross-sections can be formulated as: With strain distribution corresponding to a failure condition, find the ultimate resistances \(N, M_x, M_y\) so as to fulfil the basic equations of equilibrium and one of the following linear constraints:

\[
\begin{align*}
(a) \quad & \begin{cases} \mathcal{L}_1(N, M_x, M_y) = M_x - M_x^0 = 0 \\ \mathcal{L}_2(N, M_x, M_y) = M_y - M_y^0 = 0 \end{cases} \\
(b) \quad & \begin{cases} \mathcal{L}_1(N, M_x, M_y) = N - N_0 = 0 \\ \mathcal{L}_2(N, M_x, M_y) = M_x - M_x^0 = 0 \end{cases}
\end{align*}
\] (3)

---

Figure 3: Solution procedures. (a) Interaction diagrams for given bending moments; (b) Moment-capacity contours for given axial force and bending moment \(M_x\).
where $N_0, M_{x0}, M_{y0}$ represents the given axial force and bending moments, respectively.

An incremental-iterative procedure based on arc-length constraint equation is proposed in order to determine the biaxial strength of an arbitrary composite steel-concrete cross section accordingly to the already described situations (Fig. 3). Consider an irregular composite section as shown in Figure 1. The global $x, y$-axes of the cross section could have their origin either in elastic or plastic centroid of the cross-section. For each inclination of the neutral axis defined by the parameters $\phi_x$ and $\phi_y$, the farthest point on the compression side (or the most tensioned steel bar position) is determined (i.e. the point with co-ordinates $x_c, y_c$). We assume that at this point the failure condition is met, and consequently the axial compressive strain $\varepsilon_0$ can be expressed as:

$$\varepsilon_0 = \varepsilon_u - (\phi_y y_c + \phi_x x_c)$$

(4)

Thus, resultant strain distribution corresponding to the curvatures $\phi_x$ and $\phi_y$ can be expressed in linear form as:

$$\varepsilon(\phi_x, \phi_y) = \varepsilon_u + \phi_x (y - y_c) + \phi_y (x - x_c)$$

(5)

In this way, substituting the strain distribution given by the Eq. (5) in the basic equations of equilibrium, the unknown $\varepsilon_0$ together with the failure constraint equation can be eliminated from the nonlinear system (2). Thus the basic equations of equilibrium together with the linear constraints Eqs. (3a)-(3b) forms a determined nonlinear system of equations (i.e. 5 equations and 5 unknowns), and the solutions can be obtained iteratively following an approach outlined in the next sections. Based on Green's theorem, the integration of the stress resultant and stiffness coefficients over the cross-section will be transformed into line integrals along the perimeter of the cross-section. In order to perform the integral of a determined side of the contour, polygonal or circular, of the integration area, the interpolatory Gauss-Lobatto method is used [7].

### 2.2.1 Interaction diagrams for given bending moments

In this case introducing the constraints (3a) in the system (2) the problem of the ultimate strength analysis of cross-section can be expressed mathematically as (Fig. 3.a):

$$\int_{A} \sigma(\phi_x, \phi_y) dA - N = 0$$

$$\int_{A} \sigma(\phi_x, \phi_y) dA - \lambda M_{x0} = 0$$

$$\int_{A} \sigma(\phi_x, \phi_y) dA - \lambda M_{y0} = 0$$

(6)
in which axial load $N$ and curvatures $\phi_x$ and $\phi_y$ represents the unknowns and $\lambda$ represents the load parameter defining the intensity of the bending moments. If we regard the curvatures as independent variables in axial force equation, the curvatures and the load amplifier factor $\lambda$ are given by solving the following nonlinear system of equations:

$$\int_A \sigma(x,y) y dA - \lambda M_{x0} = 0$$
$$\int_A \sigma(x,y) x dA - \lambda M_{y0} = 0$$

(7)

This can be rewritten in terms of non-linear system of equations in the following general form:

$$F(\lambda, \Phi) = f^{\text{int}} - \lambda f^{\text{ext}} = 0$$

(8)

To traverse a solution path a proper parametrization is needed. A common setting of a continuation process is to augment the equilibrium equations (8) with a constraint [7, 8]. In this case the curvature-moment constraint can be defined by equation $g$ in the following form:

$$f^{\text{int}} - \lambda f^{\text{ext}} = 0; \quad g(\lambda, \Phi) = 0$$

(9)

In this procedure, commonly called arc-length method, these equations are solved in a series of steps or increments, usually starting from the unloaded state ($\lambda = 0$), and the solution to (9) is referred to as equilibrium path. Instead of solving Eqs. (9) directly, an indirect solution scheme for the constraint equation may be introduced. According to the indirect arc-length technique [8], the iterative changes of curvature vector $\delta \Phi$ for the new unknown load level is written as:

$$\delta \Phi = -K_T^{-1} F + \delta \lambda K_T^{-1} f^{\text{ext}} = \delta F + \delta \lambda \delta \Phi_T$$

(10)

where $F$ represents the out-of-balance force vector (Eq. 8) and $K_T$ represents the tangent stiffness matrix of the cross-section:

$$K_T = \left( \frac{\partial F}{\partial \Phi} \right) = \begin{bmatrix}
\frac{\partial M_{x}^{\text{int}}}{\partial \phi_x} & \frac{\partial M_{y}^{\text{int}}}{\partial \phi_x} \\
\frac{\partial M_{x}^{\text{int}}}{\partial \phi_y} & \frac{\partial M_{y}^{\text{int}}}{\partial \phi_y} \\
\frac{\partial M_{x}^{\text{int}}}{\partial \phi_y} & \frac{\partial M_{y}^{\text{int}}}{\partial \phi_y}
\end{bmatrix}$$

(11)

in which the partial derivatives are with respect to the strains and stresses evaluated at current iteration $k$. Assuming the strain distribution given by the Eq.(5), the coefficients of the stiffness matrix can be symbolically evaluated as:
\[ k_{ij} = \frac{\partial M_{ij}}{\partial \phi_i} = \frac{\partial}{\partial \phi_i} \int_{A} \sigma(e(\phi_i, \phi_j)) y dA = \int_{A} \frac{\partial \sigma}{\partial e} \frac{\partial e}{\partial \phi_i} y dA = \int_{A} E_y(y - y_c) dA \]

\[ k_{11} = \frac{\partial M_{11}}{\partial \phi_x} = \frac{\partial}{\partial \phi_x} \int_{A} \sigma(e(\phi_x, \phi_y)) y dA = \int_{A} \frac{\partial \sigma}{\partial e} \frac{\partial e}{\partial \phi_x} y dA = \int_{A} E_y(y - y_c) dA \]

\[ k_{12} = \frac{\partial M_{12}}{\partial \phi_y} = \frac{\partial}{\partial \phi_y} \int_{A} \sigma(e(\phi_x, \phi_y)) y dA = \int_{A} \frac{\partial \sigma}{\partial e} \frac{\partial e}{\partial \phi_y} y dA = \int_{A} E_y(y - y_c) dA \]

\[ k_{21} = \frac{\partial M_{21}}{\partial \phi_x} = \frac{\partial}{\partial \phi_x} \int_{A} \sigma(e(\phi_x, \phi_y)) x dA = \int_{A} \frac{\partial \sigma}{\partial e} \frac{\partial e}{\partial \phi_x} x dA = \int_{A} E_x(x - x_c) dA \]

\[ k_{22} = \frac{\partial M_{22}}{\partial \phi_y} = \frac{\partial}{\partial \phi_y} \int_{A} \sigma(e(\phi_x, \phi_y)) x dA = \int_{A} \frac{\partial \sigma}{\partial e} \frac{\partial e}{\partial \phi_y} x dA = \int_{A} E_x(x - x_c) dA \]

(12)

Figura 4: Diagramas de interacción para momentos de flexión dados. Flujograma de análisis.
where the coefficients $k_{ij}$ are expressed in terms of the tangent modulus of elasticity $E_t$. Thus the incremental curvatures for the next iteration can be written as:

$$\Delta \Phi_{k+1} = \Delta \Phi_k + \delta \Phi$$  \hspace{1cm} (13)

Assuming that a point ($\tau^{t}$, $\lambda^{t}$) of the equilibrium path has been reached, the next point ($\tau^{t+\Delta \tau}$, $\lambda^{t+\Delta \tau}$) of the equilibrium path is then computed updating the loading factor and curvatures as:

$$\begin{align*}
\tau^{t+\Delta \tau} &= \tau^{t} + \Delta \tau^{t+1} \\
\lambda^{t+\Delta \tau} &= \lambda^{t} + \Delta \lambda^{t+1}
\end{align*}$$  \hspace{1cm} (14)

In this way with curvatures and loading factor known, the axial force resistance $N$ is computed based on the resultant strain distribution corresponding to the curvatures $\phi_x$ and $\phi_y$ through equation (6), and the ultimate bending moments, $M_x$ and $M_y$, are obtained scaling the reference external moments $M_{x0}$ and $M_{y0}$ through current loading factor $\lambda$ given by Equation (14). As it was stated previously, the failure of the cross section can be controlled either by the most compressed concrete point with $\varepsilon_u = \varepsilon_c$ or the most tensioned steel reinforcement bar with $\varepsilon_u = \varepsilon_s$. Starting the iterative process with control in compression, the interaction diagram will be evaluated from the compression side towards tension side, whereas enforcing the failure of the cross-section in tension, at the very first iteration, the interaction diagram is evaluated from the tension to compression side. During the iterative process these controlled points are automatically interchanged. For instance, assuming that the current iterations are conducted with the most compressed point the strains profiles are defined by the same ultimate compressive strain and by different strains at the level of the most tensioned point. After the strains in the most tensioned point equal or exceed the tensile steel strain at failure, the control point becomes the most tensioned point, and the process continues similarly, but with the coordinates of this point and associated ultimate steel strain. Figure 4 shows a simplified flowchart of this analysis algorithm.

### 2.2.2 Moment capacity contour for given axial force and bending moment $M_x$

In this case, injecting the linear constraints (3b) in the nonlinear system (2), and arranging the system accordingly with the decoupled unknowns, we obtain:

$$\begin{align*}
\int dA \sigma (\varepsilon(\phi_x, \phi_y)) + M_x = 0 \\
\int dA \lambda \varepsilon_y + M_y = 0
\end{align*}$$  \hspace{1cm} (15)

in which bending moment $M_x$ and curvatures $\phi_x$ and $\phi_y$ represents the unknowns. Following a similar approach as presented above, the curvatures are obtained solving the first two equations and then with this strain distribution the bending moment resistance about $y$ axis is computed with the last equation of the system [7].
2.3 Design procedure

An iterative procedure based on Newton method is proposed to design a cross-section subjected to axial force \((N)\) and biaxial bending moments \((M_x, M_y)\). The method proposed herein consists of computing the required reinforcement \((A_{tot})\) supposing that all structural parameters are specified and, under given external loads, the cross section reach its failure either to maximum strains attained at the outer compressed point of the concrete section or to maximum strains attained in the most tensioned reinforcement steel fibre. The reinforcement layout is given as percentage \((\alpha_i)\) of the total reinforcement steel area \((A_{tot})\) in each location. The nonlinear system of equations (2) can be rewritten in this case as:

\[
\begin{align*}
\int \sigma(e_{0}, \phi_x, \phi_y) dA + A_{tot} \sum_{i=1}^{N_b} \sigma(e_i, (e_{0}, \phi_x, \phi_y)) \epsilon_i - N &= 0 \\
\int \sigma(e_{0}, \phi_x, \phi_y) dA + A_{tot} \sum_{i=1}^{N_b} \sigma(e_i, (e_{0}, \phi_x, \phi_y)) y_i \alpha_i - M_x &= 0 \\
\int \sigma(e_{0}, \phi_x, \phi_y) dA + A_{tot} \sum_{i=1}^{N_b} \sigma(e_i, (e_{0}, \phi_x, \phi_y)) x_i \alpha_i - M_y &= 0 \\
\epsilon_0 + \phi_x y_i (\phi_x, \phi_y) + \phi_y x_i (\phi_x, \phi_y) - \epsilon_a &= 0
\end{align*}
\]  

in which \(A_{tot}, e_0, \phi_x, \phi_y\) represent the unknowns, the surface integral is extended over compressive concrete and structural steel areas \((A_{cs})\) and \(N_b\) represents the number of steel reinforcements bars. As in the previous cases, substituting the strain distribution given by the Eq. (5) in the basic equations of equilibrium, the unknown \(\epsilon_0\) together with the failure constraint equation can be eliminated from the nonlinear system (16). Thus, the nonlinear system of equations (16) is reduced to an only three basic equations of equilibrium as:

\[
\begin{align*}
\int \sigma(e_{0}, \phi_x, \phi_y) dA + A_{tot} \sum_{i=1}^{N_b} \sigma(e_i, (e_{0}, \phi_x, \phi_y)) \epsilon_i - N &= 0 \\
\int \sigma(e_{0}, \phi_x, \phi_y) dA + A_{tot} \sum_{i=1}^{N_b} \sigma(e_i, (e_{0}, \phi_x, \phi_y)) y_i \alpha_i - M_x &= 0 \\
\int \sigma(e_{0}, \phi_x, \phi_y) dA + A_{tot} \sum_{i=1}^{N_b} \sigma(e_i, (e_{0}, \phi_x, \phi_y)) x_i \alpha_i - M_y &= 0
\end{align*}
\]  

in which the unknowns, total reinforcement steel area \(A_{tot}\), and the ultimate sectional curvatures \(\phi_x\) and \(\phi_y\) can be obtained iteratively following the Newton method. In this respect, the system (17) can be rewritten in terms of non-linear system of equations in the following general form:

\[
F(X) = f^{int} - f^{ext} = 0
\]  

where the external biaxial loading vector is:
For all iterations

Start with the given axial force \( N \) and bending moments \( M_x \) and \( M_y \). Initial approximation for \( A_{tot} = 0.005 A_g \).

Set the curvatures \( \phi_x \) and \( \phi_y \) to zero and select the failure mode at the very first iteration. \( k=0 \).

Compute/Update the Jacobian \( F' \) using Eq. (22).

Determine the unknowns vector iteratively as:

\[
X^{k+1} = X^k - F'(X^k)^{-1} F(X^k)
\]

Eq. (21)

Determine the difference between external and internal vectors Eq. (18).

Convergence ?

No \((k=k+1)\)

Update the strain profile and determine the control point.

Yes

Print the ultimate curvatures \( \phi_x \) and \( \phi_y \) and the total reinforcement steel area \( A_{tot} \).

Figure 5: Flowchart for design procedure.

\[
f^{ext} = \begin{bmatrix} N & M_x & M_y \end{bmatrix}^T
\]

and the internal forces vector, computed as function of the curvatures and total reinforcement steel area \( A_{tot} \), \( X = [\phi_x, \phi_y, A_{tot}]^T \) is:

\[
f^{int} = \begin{bmatrix}
N^{int} = \int_{A_{tot}} \sigma(x, y) dA + A_{tot} \sum_{i=1}^{N} \sigma(\epsilon_i, \phi_x, \phi_y) x_i \\
M_x^{int} = \int_{A_{tot}} \sigma(x, y) y dA + A_{tot} \sum_{i=1}^{N} \sigma(\epsilon_i, \phi_x, \phi_y) x_i, \alpha_i \\
M_y^{int} = \int_{A_{tot}} \sigma(x, y) x dA + A_{tot} \sum_{i=1}^{N} \sigma(\epsilon_i, \phi_x, \phi_y) y_i, \alpha_i
\end{bmatrix}
\]

Eq. (20)

According to the Newton iterative method, the iterative changes of unknowns vector \( X \) can be written as:

\[
X^{k+1} = X^k - F'(X^k)^{-1} F(X^k), \quad k \geq 0
\]
where $\mathbf{F}'$ represents the Jacobian of the nonlinear system (37) and can be expressed as:

$$
\mathbf{F}' = \frac{\partial \mathbf{F}}{\partial \mathbf{X}} = \begin{bmatrix}
\frac{\partial N_{\text{int}}}{\partial \phi_x} & \frac{\partial N_{\text{int}}}{\partial \phi_y} & \frac{\partial N_{\text{int}}}{\partial A_{\text{tot}}} \\
\frac{\partial M_{\text{int}}}{\partial \phi_x} & \frac{\partial M_{\text{int}}}{\partial \phi_y} & \frac{\partial M_{\text{int}}}{\partial M_x} \\
\frac{\partial M_{\text{int}}}{\partial \phi_x} & \frac{\partial M_{\text{int}}}{\partial \phi_y} & \frac{\partial M_{\text{int}}}{\partial M_y}
\end{bmatrix}
$$

(22)

Explicitly the expressions of the Jacobian’s coefficients are:

$$
\frac{\partial N_{\text{int}}}{\partial \phi_x} = \int E_i (y-y_c) dA + A_{\text{tot}} \sum_{i=1}^{N_i} E_i (y_i-y_c) \alpha_i
$$

$$
\frac{\partial N_{\text{int}}}{\partial \phi_y} = \int E_i (x-x_c) dA + A_{\text{tot}} \sum_{i=1}^{N_i} E_i (x_i-x_c) \alpha_i
$$

$$
\frac{\partial N_{\text{int}}}{\partial A_{\text{tot}}} = \sum_{i=1}^{N_i} \sigma(\epsilon_i) \alpha_i
$$

$$
\frac{\partial M_{\text{int}}}{\partial \phi_x} = \int E_i (y-y_c) dA + A_{\text{tot}} \sum_{i=1}^{N_i} E_i y_i (y_i-y_c) \alpha_i
$$

$$
\frac{\partial M_{\text{int}}}{\partial \phi_y} = \int E_i (x-x_c) dA + A_{\text{tot}} \sum_{i=1}^{N_i} E_i y_i (x_i-x_c) \alpha_i
$$

$$
\frac{\partial M_{\text{int}}}{\partial A_{\text{tot}}} = \sum_{i=1}^{N_i} \sigma(\epsilon_i) y_i \alpha_i
$$

$$
\frac{\partial M_{\text{int}}}{\partial \phi_x} = \int E_i (y-y_c) dA + A_{\text{tot}} \sum_{i=1}^{N_i} E_i x_i (y_i-y_c) \alpha_i
$$

$$
\frac{\partial M_{\text{int}}}{\partial \phi_y} = \int E_i (x-x_c) dA + A_{\text{tot}} \sum_{i=1}^{N_i} E_i x_i (x_i-x_c) \alpha_i
$$

(23)

These coefficients are expressed in terms of the tangent modulus of elasticity $E_i$, total reinforcement steel area $A_{\text{tot}}$ and the coordinates $x_c$, $y_c$ of the “constrained” point. As already mentioned, during the iterative process, for each inclination of the neutral axis defined by the current curvatures, $\phi_x$ and $\phi_y$, the coordinates of the constrained point can be always determined and consequently the stiffness matrix coefficients can be evaluated. The iterative procedure starts with curvatures $\phi_x=0$, $\phi_y=0$. The initial approximation for $A_{\text{tot}}$ is chosen by the user of the computer program, and represents the only parameter that controls the stability of convergence process. If the initial value of $A_{\text{tot}}$ is taken to be 0.005$A_g$ (i.e. $A_g$ is gross cross-sectional area) it is observed that the iterative process described above has converged for many different problems in six to seven iterations. Figure 5 shows a simplified flowchart of this analysis algorithm.
3 Computational examples

3.1 Example 1: Composite steel-concrete cross-section

The composite steel-concrete cross-section depicted in Fig. 6a, consists of the concrete matrix, fifteen reinforcement bars of diameter 18 mm, a structural steel element and a circular opening. Characteristic strengths for concrete, structural steel and reinforcement bars are $f'_c=30$ Mpa, $f_{st}=355$ Mpa and $f_s=460$ Mpa, respectively. These characteristic strengths are reduced by dividing them with the corresponding safety factors $\gamma_c=1.50$, $\gamma_{st}=1.10$ and $\gamma_s=1.15$. The stress-strain curve for concrete which consists of a parabolic and linear-horizontal part was used in the calculation, with the crushing strain $\varepsilon_0=0.002$ and ultimate strain $\varepsilon_{cu}=0.0035$. The Young modulus for all steel sections was 200 GPa while the maximum strain was $\varepsilon_u=\pm1\%$. The strain softening effect for the concrete in compression is taken into account, in the present approach, through the parameter $\gamma$. This is an example proposed and analysed by Chen et al. [3] and later studied by Rosati et.al [6] and others. The moment-capacity contours ($M_x$-$M_y$ interaction curve) have been determined under a given axial load $N=4120$ kN and compared with the results reported in [3], Fig. 6b. As it can be seen the results are in close agreement when the bending moments are computed about the plastic centroidal axes of the cross-section. However, the method proposed by Chen [3] does not generate genuinely plane moment-capacity curves, because of convergence problems caused by the fixed axial force. On the contrary, with the proposed approach, the interaction curve can be computed without any convergence difficulties; a maximum of three iterations are necessary to establish the equilibrium, even when the geometric centroid has been chosen as origin of the reference axes or the strain-softening of the concrete in compression is taken into account ($\gamma=0.15$).
The moment-capacity curves for these situations are also depicted in the Fig. 6b. For these cases, the reference [3] does not present comparative results. In order to verify the stability of the proposed method a series of analyses have been conducted to determine the interaction curves for different values of bending moment’s ratio $M_y/M_x = \tan(\alpha)$. The bending moments are computed about the geometric centroidal axes of the cross-section and the strain-softening of the concrete in compression has been modeled ($\gamma=0.15$). Figure 7 shows the interaction diagrams for $\alpha = 0^0, 30^0, 60^0, 90^0$. No convergence problems have been encountered using the proposed approach; a maximum of three iterations have been required to complete the entire interaction diagram in each case. This comparison illustrates the efficiency of the proposed approach and convergence stability.

Let us consider this cross-section subjected to biaxial bending to carry the following design loads: $N=4120$ kN, $M_x=210.5$ kN and $M_y=863.5$ kNm. The distribution of the steel reinforcing bars are shown in Fig. 4, and we consider that all rebars have the same diameter. The procedure described at section 2.3 is used to find the required total steel reinforcement for the cross-section to achieve an adequate resistance for the design loads. As shown in Table 1 the iterative process of design was started with $\phi_x=0$ and $\phi_y=0$, and $A_{tot}=0.005 A_g = 215$ cm$^2$. The equilibrium tolerance has been taken as $1E-4$. After only five iterations, the total area required of the rebars was found to be $A_{tot}=34.347$ cm$^2$. Consequently, the required diameter of
the selected rebar is $\Phi_{req} = 2 \sqrt{ \frac{A_{\text{tot}}}{N_f \pi} } = 1.71\text{cm}$ which compare very well with the required bar diameter reported by Chen et.al. [3], $\Phi_{req}=1.78\text{cm}$. Reinforcement bars of diameter 18 mm are thus suitable for this cross-section. In the above computational example, the axial force and bending moments are represented about the plastic centroidal axes of the cross-section. When the design loads are represented about the geometric centroid the total area required of the rebars was found to be $A_{\text{tot}}=15.674 \text{ cm}^2$. This result has been obtained after only six iterations. For this case the reference [3] does not present comparative results.

3.2 Example 2: Design of bi-axially loaded box cross-section

The box cross-section, depicted in Figure 8.a, consists of the concrete matrix and sixteen reinforcement bars, all rebars having the same diameter. This section is subjected to the following design loads: $N=2541.7$ kN, $M_x=645.6$ kNm, and $M_y=322.8$ kNm. Characteristic strengths for concrete and reinforcement bars are: $f'_c = 23.443\text{MPa}, f_y = 413.69 \text{MPa}$ respectively.

![Figure 8](image1.png)

Figure 8: (a) Bi-axially loaded box cross-section; (b) Plastic status of section under design loads and $A_{\text{tot}}=40.586 \text{ cm}^2$.

The stress-strain curve for concrete which consists of a parabolic and linear-descending part was used in the calculation, with the crushing strain $\varepsilon_{cr}=0.002$ and ultimate strain $\varepsilon_{cu}=0.0038$. The Young modulus for reinforcing bars was 200GPa while the maximum strain was $\varepsilon_u=\pm1\%$. The strain softening effect for the concrete in compression is taken into account, in the present approach, through the parameter $\gamma$. This problem was also solved by Rodrigues & Ochoa [2]. The procedure described at section 2.3 is used to find the required total steel reinforcement for the cross-section to achieve an adequate resistance for the above mentioned design loads. The strain softening parameter $\gamma=0.15$ for the concrete in compression has been considered in analysis.
Table 2: Main parameters involved in the iterative process.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$\phi_x$</th>
<th>$\phi_y$</th>
<th>$A_{tot}$ [cm²]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>0.000</td>
<td>0.000</td>
<td>0.005A_g=12.25</td>
<td>1.000</td>
</tr>
<tr>
<td>1</td>
<td>1.089E-4</td>
<td>5.576E-5</td>
<td>115.41</td>
<td>1.000146</td>
</tr>
<tr>
<td>2</td>
<td>6.426E-5</td>
<td>5.264E-5</td>
<td>30.816</td>
<td>0.599621</td>
</tr>
<tr>
<td>3</td>
<td>7.102E-5</td>
<td>4.550E-5</td>
<td>39.682</td>
<td>0.173854</td>
</tr>
<tr>
<td>4</td>
<td>7.084E-5</td>
<td>4.559E-5</td>
<td>40.556</td>
<td>0.007475</td>
</tr>
<tr>
<td>5</td>
<td>7.084E-5</td>
<td>4.5583E-5</td>
<td>40.585</td>
<td>0.000266</td>
</tr>
<tr>
<td>6</td>
<td>7.08345E-5</td>
<td>4.5583E-5</td>
<td>40.586</td>
<td>0.0000856</td>
</tr>
</tbody>
</table>

As shown in Table 2 the iterative process of design was started with $\phi_x=0$ and $\phi_y=0$, and $A_{tot}=0.005A_g=12.25$ cm². The equilibrium tolerance has been taken as 1E-4. After only six iterations, the total area required of the rebars was found to be $A_{tot}=40.586$ cm². This means that the area required of the selected rebar is 2.536 cm² which compare very well with the required area, 2.535 cm², reported by Rodrigues & Ochoa [2]. Figure 8.b shows the plastic status of the cross-section associated to the equilibrium between external design loads and internal forces with the total reinforcing area obtained after six iterations. Figure 9 shows the corresponding interaction curves for both $M_x$ and $M_y$ of this section for $\alpha=26.56^\circ$. Figure 10 shows the moment capacity diagrams for $N=2541.7$ kN considering different levels of confinement. Rebars of diameter $3/4''\approx 1.90$ cm has been considered in these analyses. As it can be seen, by reducing the confinement in the concrete ($\gamma=0.15, 0.50$) the interaction curves indicate lower capacities (Fig. 10) and the non-convexity of the diagrams is more pronounced, and also, near the compressive axial load capacity multiple solutions exist in the $N-M$ space when the strain softening is modelled, $\gamma=0.15$ (Fig. 9) [7].

Figure 9: Biaxial interaction curves for box cross-section $\alpha=26.56^\circ$. 
The effects of confinement and creep in the concrete over the total steel reinforcing area are presented in Tables 3 and 4. The section is subjected to the same design loads as in the previously computation. As it can be seen in Table 3 by reducing the confinement in the concrete (i.e. by increasing the value of $\gamma$ in descending branch of compressed concrete stress-strain curve) the total reinforcing area increases. No convergence problems have been experienced by the proposed approach, a maximum of seven iterations have been required to determine the reinforcement area for the case when $\gamma=0.5$.

![Figure 10: Moment capacity contours of box cross-section with compressive axial force $N=2541.7$ kN.](image)

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$A_{tot}$ [cm²]</th>
<th>No.of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>37.521</td>
<td>5</td>
</tr>
<tr>
<td>0.15</td>
<td>40.586</td>
<td>6</td>
</tr>
<tr>
<td>0.30</td>
<td>43.788</td>
<td>7</td>
</tr>
<tr>
<td>0.50</td>
<td>48.428</td>
<td>7</td>
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</table>

Table 3: Confinement effect over $A_{tot}$

<table>
<thead>
<tr>
<th>$\varepsilon_{0}$ ($\varepsilon_{0}=1.9\varepsilon_{0}$)</th>
<th>$A_{tot}$ [cm²]</th>
<th>No.of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.002</td>
<td>40.586</td>
<td>6</td>
</tr>
<tr>
<td>0.003</td>
<td>36.307</td>
<td>6</td>
</tr>
<tr>
<td>0.004</td>
<td>35.189</td>
<td>6</td>
</tr>
<tr>
<td>0.005</td>
<td>34.578</td>
<td>6</td>
</tr>
<tr>
<td>0.007</td>
<td>34.115</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 4: Creep effect over $A_{tot}$
The effect of creep of the concrete has been investigated by varying the value of crushing strain $\varepsilon_{c0}$. The ultimate strain in the compressed concrete is considered as: $\varepsilon_{cu} = 1.90\varepsilon_{c0}$. As shown in Table 4 the total reinforcement steel area decreases in amount as the concrete creep increases. No convergence problems have been experienced, in all situations, the iterative procedure has converged in only six iterations.

4 Conclusions

A new computer method has been developed for the rapid design and ultimate strength capacity evaluation of composite steel-concrete cross-sections subjected to axial force and biaxial bending.

Comparing the algorithm developed in this paper, for ultimate strength capacity evaluation, with the existing methods, it can be concluded that the proposed approach is general, can determine both interaction diagrams and moment capacity contours, and, of great importance, it is fast, the diagrams are directly calculated by solving, at a step, just two coupled nonlinear equations.

The iterative algorithm, proposed herein, to design the steel reinforcement of composite steel-concrete sections under biaxial bending and axial force, as compared to other iterative methods, is very stable, converging rapidly if the initial value of the required reinforcement area is properly selected. Convergence is assured for any load case, even near the state of pure compression or tension and is not sensitive to the initial/starting values, to how the origin of the reference loading axes is chosen or to the strain softening effect for concrete in compression.

Using the proposed method we have found that near the axial load capacity under pure compression, when the strain-softening of the concrete is taken into account, the solution is not unique which implies non-convexity of the failure surface in these situations. Therefore, the proposed approach based on arc-length constraint strategy is essential to assure the convergence of the entire process and to determine all possible solutions.

The method has been verified by comparing the predicted results with the established results available from the literature. It can be concluded that the proposed numerical method proves to be reliable and accurate for practical applications in the design of composite steel-concrete beam-columns and can be implemented in the advanced analysis techniques of 3D composite frame structures.

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References


