PROBLEME DE VALORI PROPRII

1. FORMULAREA PROBLEMEI DE VALORI SI VECTORI PROPRII

Problema de valori proprii se defineste astfel: *fiind data o matrice **A** de dimesniuni n x n, λ se numeste valoare proprie a matricei A si* $\mathbf{X} \neq \mathbf{0}$ *vector propriu asociat, daca este satisfacuta relatia:*

$$\mathbf{AX} = \lambda \mathbf{X} \tag{1}$$

Se obtine

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{0} \tag{2}$$

unde **I** este matricea unitate. Aceasta este forma standard a problemei de valori proprii. Considerind matricea **A**, respectiv matricea unitate **I**:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}; \ \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{3}$$

si $\mathbf{X}^T = (x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n)$ suntem condusi la urmatorul sistem de ecuatii liniar si omogen cu solutia nebanala ($\mathbf{X} \neq \mathbf{0}$):

$$\begin{cases} (a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0 \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)x_3 + \dots + a_{3n}x_n = 0 \\ \dots. \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + (a_{nn} - \lambda)x_n = 0 \end{cases} \tag{4}$$

Rezulta ca determinantul matricei sistemului ($\mathbf{A} - \lambda \mathbf{I}$) trebuie sa fie nul, adica:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} - \lambda \end{vmatrix} = 0 \tag{5}$$

ceea ce inseamna ca matricea ($\mathbf{A} - \lambda \mathbf{I}$) este singulara. Conditia se mai scrie:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{6}$$

Dezvoltind determinantul se obtine o ecuatie algebrica, de gradul *n* in λ de forma:

$$P(\lambda) = 0 \tag{7}$$

in care $P(\lambda)$ se numeste polinom carcateristic, iar ecuatia asociata se numeste ecuatie caracteristica sau ecuatie seculara. Rezolvarea ecuatiei carcateristice conduce la valorile proprii ale matricei **A**, considerate distincte si care se ordoneaza:

$$|\lambda_1| > |\lambda_2| > ... > |\lambda_n| \tag{8}$$

Vectorul propriu $\mathbf{X}_j$, j=1,2,...,n, asociat valorii proprii λ_j, se determina in felul urmator: se inlocuieste in sistemul de ecuatii (4), λ cu valoarea λ_j, sistemul fiind nedeterminat o data, se da unei dintre necunoscute (de regula lui x_1 sau lui x_n) valoare arbitrara (de regula valoarea 1), din sistem considerind in continuare (n-1) ecuatii cu (n-1) necunoscute, care poate fi rezolvat cu una dintre metodele numerice de rezolvare a sistemelor de ecuatii algebrice liniare si neomeogene. Se obtin astfel perechile:

$$(\lambda_1, \mathbf{X}_1);\ (\lambda_2, \mathbf{X}_2);\ ...;\ (\lambda_n, \mathbf{X}_n) \tag{9}$$

unde:

$$\mathbf{X}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ ... \\ x_{n1} \end{bmatrix};\ \mathbf{X}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \\ ... \\ x_{n2} \end{bmatrix};\ ....;\ \mathbf{X}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ x_{3n} \\ ... \\ x_{nn} \end{bmatrix} \tag{10}$$

sunt vectorii proprii. Perechea $(\lambda_j, \mathbf{X}_j)$ defineste modul "j" al matricei **A**. Acesti vectori proprii pusi unul linga altul formeaza matricea spectrala **S**:

$$\mathbf{S} = \begin{bmatrix} x_{11} & x_{12} & ... & x_{1j} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2j} & ... & x_{2n} \\ x_{31} & x_{32} & ... & x_{3j} & ... & x_{3n} \\ . & . & . & . & . & . \\ x_{n1} & x_{n2} & ... & x_{nj} & ... & x_{nn} \end{bmatrix} \tag{11}$$

O matrice oarecare **A** reala de dimesniuni n x n, poate avea valorile proprii reale sau complexe, vectorii proprii pot avea si ei elemente reale sau complexe.

2. PROPRIETATI ALE VALORILOR SI VECTORILOR PROPRII IN CAZUL MATRICELOR REALE SIMETRICE SI POZITIV DEFINITE

Proprietatea 1. O matrice reala simetrica are valorile proprii reale si vectorii proprii ortogonali.

Pentru a demonstra ca o matrice reala simetrica are valorile proprii reale, presupunem prin reducere la absurd ca valorile proprii sunt complexe. Cum polinomul caracteristic are coeficientii reali rezulta ca ecuatia caracteristica $P(\lambda) = 0$, $\lambda \in \mathbf{C}$ va admite ca radacina si pe conjugata $\bar{\lambda} \in \mathbf{C}$. Valorii proprii λ ii corespunde vectorul propriu **X**, iar valorii proprii $\bar{\lambda}$ vectorul propriu $\bar{\mathbf{X}}$. Astfel putem scrie:

$$\begin{aligned} \mathbf{AX} &= \lambda \mathbf{X} \\ \mathbf{A\overline{X}} &= \overline{\lambda}\mathbf{\overline{X}} \end{aligned} \tag{12}$$

Inmultind prima relatie (12) la stinga cu $\overline{\mathbf{X}}^T$, iar a doua cu $\mathbf{X}^T$ la stinga obtinem:

$$\overline{\mathbf{X}}^T\mathbf{AX} = \lambda\overline{\mathbf{X}}^T\mathbf{X} \tag{13}$$

$$\mathbf{X}^T\mathbf{A}\overline{\mathbf{X}} = \overline{\lambda}\mathbf{X}^T\overline{\mathbf{X}} \tag{14}$$

Dar cum $\overline{\mathbf{X}}^T\mathbf{AX} = \left(\mathbf{X}^T\mathbf{A}\overline{\mathbf{X}}\right)^T = \overline{\mathbf{X}}^T\mathbf{A}^T\left(\mathbf{X}^T\right)^T$ si scazind relatiile (13,14) membru cu membru, gasim:

$$0 = \left(\lambda - \overline{\lambda}\right)\mathbf{X}^T\overline{\mathbf{X}} \tag{15}$$

sau ca $\lambda = \overline{\lambda}$, $\mathrm{Im}(\lambda) = 0$; $\lambda \in \mathbf{R}$. S-a tinut cont ca $\overline{\mathbf{X}}^T\mathbf{X} = \mathbf{X}^T\overline{\mathbf{X}}$, care reprezinta produsul scalar a doi vectori, deci este comutativ.

Pentru demonstrarea ortogonalitatii a doi vectori proprii sa consideram vectorii proprii $\mathbf{X}_i$ si $\mathbf{X}_j$ care corespund la doua valori proprii distincte λ_i si λ_j. Avind in vedere relatiile evidente:

$$\begin{aligned} \mathbf{AX}_i &= \lambda_i\mathbf{X}_i \\ \mathbf{AX}_j &= \lambda_j\mathbf{X}_j \end{aligned} \tag{16}$$

si inmultind prima relatie la stinga cu $\mathbf{X}_j^T$ iar pe a doua cu $\mathbf{X}_i^T$ obtinem:

$$\begin{aligned} \mathbf{X}_j^T\mathbf{AX}_i &= \lambda_i\mathbf{X}_j^T\mathbf{X}_i \\ \mathbf{X}_i^T\mathbf{AX}_j &= \lambda_j\mathbf{X}_i^T\mathbf{X}_j \end{aligned} \tag{17}$$

Avind in vedere faptul ca rezultatele acestor produse sunt scalari egali si scazind membru cu membru relatiile de mai sus obtinem:

$$\left(\lambda_i - \lambda_j\right)\mathbf{X}_j^T\mathbf{X}_i = 0 \tag{18}$$

Cum valorile proprii s-au presupus distincte rezulta ca $\mathbf{X}_j^T\mathbf{X}_i = 0$. Ortogonalitatea vectorilor se va putea exprima prin:

$$\mathbf{X}_j^T\mathbf{X}_i = \delta_{ij} = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases} \tag{19}$$

S-au considerat vectorii normati prin:

$$\mathbf{X}_j^T\mathbf{X}_j = 1 \tag{20}$$

sau scris detaliat:

$$x_{1j}^2 + x_{2j}^2 + \ldots + x_{nj}^2 = 1 \tag{21}$$

Astfel matricea spectrala **S**, este ortogonala (ortonormata) $\mathbf{S}^T\mathbf{S} = \mathbf{I}$:

$$\mathbf{S}^T \cdot \mathbf{S} = \begin{bmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{12} & x_{22} & \ldots & x_{n2} \\ . & . & . & . \\ x_{1n} & x_{2n} & \ldots & x_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ . & . & . & . \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1^T\mathbf{X}_1 & \mathbf{X}_1^T\mathbf{X}_2 & \ldots & \mathbf{X}_1^T\mathbf{X}_n \\ \mathbf{X}_2^T\mathbf{X}_1 & \mathbf{X}_2^T\mathbf{X}_2 & \ldots & \mathbf{X}_2^T\mathbf{X}_n \\ . & . & . & . \\ \mathbf{X}_n^T\mathbf{X}_1 & \mathbf{X}_n^T\mathbf{X}_2 & \ldots & \mathbf{X}_n^T\mathbf{X}_n \end{bmatrix} = \mathbf{I} \tag{22}$$

Proprietatea 2: O matrice reala simetrica poate fi diagonalizata.

Astfel $\mathbf{S}^T\mathbf{AS} = \mathbf{\Lambda}$, in care $\mathbf{\Lambda}$ este o matrice diagonala si are ca elemente valorile proprii ale matricei **A**. Produsul **AS** se scrie:

$$\mathbf{AS} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \dots & \mathbf{X}_n \end{bmatrix} = \begin{bmatrix} \mathbf{AX}_1 & \mathbf{AX}_2 & \dots & \mathbf{AX}_n \end{bmatrix} =$$
$$= \begin{bmatrix} \lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \dots & \lambda_n\mathbf{X}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} & \dots & \lambda_n x_{1n} \\ \lambda_1 x_{21} & \lambda_2 x_{22} & \dots & \lambda_n x_{2n} \\ . & . & . & . \\ \lambda_1 x_{n1} & \lambda_2 x_{n2} & \dots & \lambda_n x_{nn} \end{bmatrix} \tag{23}$$

$$\mathbf{S}^T\mathbf{AS} = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{12} & x_{22} & \dots & x_{n2} \\ . & . & . & . \\ x_{1n} & x_{2n} & \dots & x_{nn} \end{bmatrix} \cdot \mathbf{AS} = \begin{bmatrix} \lambda_1\mathbf{X}_1^T\mathbf{X}_1 & \lambda_2\mathbf{X}_1^T\mathbf{X}_2 & \dots & \lambda_n\mathbf{X}_1^T\mathbf{X}_n \\ \lambda_1\mathbf{X}_2^T\mathbf{X}_1 & \lambda_2\mathbf{X}_2^T\mathbf{X}_2 & \dots & \lambda_n\mathbf{X}_2^T\mathbf{X}_n \\ . & . & . & . \\ \lambda_1\mathbf{X}_n^T\mathbf{X}_1 & \lambda_2\mathbf{X}_n^T\mathbf{X}_2 & \dots & \lambda_n\mathbf{X}_n^T\mathbf{X}_n \end{bmatrix} =$$
$$= \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = \mathbf{\Lambda} \tag{24}$$

Spunem ca prin transformarea ortogonala $\mathbf{S}^T\mathbf{AS}$**,** matricea **A** este diagonalizata, sau adusa la forma diagonala, observindu-se ca pe diagonala principala avem chiar valorile proprii ale matricei **A.**
Din

$$\mathbf{S}^T\mathbf{AS} = \mathbf{\Lambda} \tag{25}$$

rezulta

$$\mathbf{SS}^T\mathbf{ASS}^T = \mathbf{S\Lambda S}^T \tag{26}$$

respectiv

$$\mathbf{A} = \mathbf{S\Lambda}S^T \tag{27}$$

Observatie:
Punerea matricei **A** sub forma $\mathbf{A}=\mathbf{S\Lambda S}^T$, este o factorizare a matricei A, putind fi utilizata in rezolvarea sistemului de ecuatii liniare care au matricea simetrica.

Proprietatea 3: O matrice reala simetrica pozitiv definita are valorile proprii pozitive.

Matricea **A** fiind pozitiv definita, avem:

$$\mathbf{X}^T\mathbf{A}\mathbf{X} > 0,\ \ \forall \mathbf{X} \neq \mathbf{0} \tag{28}$$

Aceasta devine:

$$\mathbf{X}^T\mathbf{A}\mathbf{X} = \mathbf{X}^T\mathbf{S}\boldsymbol{\Lambda}S^T\mathbf{X} \tag{29}$$

iar cu notatia

$$\mathbf{y} = \mathbf{S}^T\mathbf{X};\ \ \mathbf{y}^T = \mathbf{X}^T\mathbf{S} \tag{30}$$

obtinem:

$$\mathbf{y}^T\boldsymbol{\Lambda}\mathbf{y} > 0,\ \ \forall \mathbf{y} \neq \mathbf{0} \tag{31}$$

care mai poate fi scris in forma detaliata astfel:

$$\begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} > 0 \tag{32}$$

respectiv:

$$\sum_{i=1}^{n} \lambda_i y_i^2 > 0,\ \ \forall y_i \neq 0 \tag{33}$$

relatie care este indeplinita daca

$$\lambda_i > 0,\ \ i = 1,2,\dots,n \tag{34}$$

adica daca valorile proprii ale matricei **A** sunt pozitive.

Observatie:
Este valabila si reciproca: daca o matrice **A** simetrica are valorile proprii reale si pozitive ea este pozitiv definita.

Proprietatea 4: Daca λ este o valoare proprie a matricei, atunci λ^k este o valoare proprie a matricei $\mathbf{A}^k$.

Din relatia care defineste problema de valori proprii avem:

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{X} \tag{35}$$

deducem:

$$\begin{aligned} \mathbf{A}^2\mathbf{X} &= \lambda\mathbf{A}\mathbf{X} = \lambda^2\mathbf{X} \\ \mathbf{A}^3\mathbf{X} &= \lambda^2\mathbf{A}\mathbf{X} = \lambda^3\mathbf{X} \\ &\dots. \\ \mathbf{A}^n\mathbf{X} &= \lambda^{n-1}\mathbf{A}\mathbf{X} = \lambda^n\mathbf{X} \end{aligned} \tag{36}$$

Faptul ca matricea simetrica $\mathbf{A}^k$ are valorile proprii λ^k mai poate fi demonstrat si dupa cum urmeaza. Din relatia care exprima diagonalizarea matricei **A**, vom putea scrie:

$$\begin{aligned}
&\mathbf{S}^T\mathbf{A}\mathbf{S}=\mathbf{\Lambda}\\
&\left(\mathbf{S}^T\mathbf{A}\mathbf{S}\right)\left(\mathbf{S}^T\mathbf{A}\mathbf{S}\right)=\mathbf{\Lambda}^2\\
&\mathbf{S}^T\mathbf{A}\mathbf{S}\mathbf{S}^T\mathbf{A}\mathbf{S}=\mathbf{\Lambda}^2\\
&\mathbf{S}^T\mathbf{A}^2\mathbf{S}=\mathbf{\Lambda}^2\\
&\left(\mathbf{S}^T\mathbf{A}^2\mathbf{S}\right)\left(\mathbf{S}^T\mathbf{A}\mathbf{S}\right)=\mathbf{\Lambda}^3\\
&\mathbf{S}^T\mathbf{A}^2\mathbf{S}\mathbf{S}^T\mathbf{A}\mathbf{S}=\mathbf{\Lambda}^3\\
&\mathbf{S}^T\mathbf{A}^3\mathbf{S}=\mathbf{\Lambda}^3\\
&\ldots\\
&\mathbf{S}^T\mathbf{A}^n\mathbf{S}=\mathbf{\Lambda}^n
\end{aligned}\tag{37}$$

In baza relatiilor de mai sus suntem condusi la aceeasi concluzie, adica daca matricea **A** are valorile proprii λ, matricea $\mathbf{A}^2$ are valorile proprii λ^2 etc., diagonalizarea extinzindu-se si asupra matricelor $\mathbf{A}^2$, $\mathbf{A}^3$, ...

Proprietatea 5: Urma matrcei **A** *este:* $\mathrm{Tr}(\mathbf{A})=\sum_{i=1}^{n}a_{ii}=\sum_{i=1}^{n}\lambda_i$. *Urma puterii unei matrice este egala cu suma puterilor valorilor proprii:*

$$\mathrm{Tr}\left(\mathbf{A}^{\mathrm{k}}\right)=\sum_{i=1}^{n}\lambda_i^k \tag{38}$$

Dezvoltind determinantul

$$P(\lambda)=\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22}-\lambda & \ldots & a_{2n}\\ . & . & . & .\\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} \tag{39}$$

ecuatia caracteristica va fi:

$$\lambda^n-\left(a_{11}+a_{22}+\ldots+a_{nn}\right)\lambda^{n-1}+\ldots=0 \tag{40}$$

sau cu definitia data anterior:

$$\lambda^n-\mathrm{Tr}(\mathbf{A})\lambda^{n-1}+\ldots=0 \tag{41}$$

Avind in vedere relatiile Newton-Viete intre radacinile si coeficientii unei ecuatii algebrice, rezulta:

$$\lambda_1+\lambda_2+\ldots+\lambda_n=\mathrm{Tr}(\mathbf{A}) \tag{42}$$

Teorema Cayley-Hamilton. Orice matrice simetrica si patrata isi satisface propriul polinom caracteristic. Daca $P(\lambda)=0$ *atunci si* $P(\mathbf{A})=\mathbf{0}$.

$$\begin{aligned}
&P(\lambda)=\lambda^n+d_1\lambda^{n-1}+\ldots+d_{n-1}\lambda+d_n=0\\
&P(\mathbf{A})=\mathbf{A}^n+d_1\mathbf{A}^{n-1}+\ldots+d_{n-1}\mathbf{A}+d_n\mathbf{I}=\mathbf{0}
\end{aligned}\tag{43}$$

Tinind cont ca

$$A^k = \mathbf{S}\mathbf{\Lambda}^k\mathbf{S}^T, \;\; k = 1,2,\ldots,n \tag{44}$$

astfel ca

$$P(\mathbf{A}) = \mathbf{S}\mathbf{\Lambda}^n\mathbf{S}^T + d_1\mathbf{S}\mathbf{\Lambda}^{n-1}\mathbf{S}^T + \ldots + d_{n-1}\mathbf{S}\mathbf{\Lambda}\mathbf{\Lambda}^T + d_n\mathbf{S}\mathbf{I}\mathbf{S}^T = \mathbf{0} \tag{45}$$

sau

$$P(\mathbf{A}) = \mathbf{S}\left(\mathbf{\Lambda}^n + d_1\mathbf{\Lambda}^{n-1} + \ldots + d_{n-1}\mathbf{\Lambda} + d_n\mathbf{I}\right)\mathbf{S}^T = \mathbf{0} \tag{46}$$

Scriem dezvoltat si obtinem:

$$P(\mathbf{A}) = \mathbf{S}\begin{bmatrix} P(\lambda_1) & 0 & \ldots & 0 \\ 0 & P(\lambda_2) & \ldots & 0 \\ . & . & . & 0 \\ 0 & 0 & \ldots & P(\lambda_n) \end{bmatrix}\mathbf{S}^T = \mathbf{0} \tag{47}$$

deoarece $P(\lambda_j) = 0, \;\; j = 1,2,\ldots,n.$

Observatie:
Valabilitatea acestei teoreme se poate extinde si la matrice patrate oarecare.

3. METODA PUTERII PENTRU CALCULUL VALORILOR SI VECTORILOR PROPRII

Metoda puterii sau metoda iterarii matriceale (elaborata de Von Mises) pentru calculul valorilor si vectorilor proprii ale unei matrice **A**, este o metoda care genereaza siruri de iterate ale valorilor proprii si ale vectorilor proprii, fiind bazata pe relatia de definitie a problemei de valori si vectori proprii a matricei **A**:

$$\mathbf{AX} = \lambda\mathbf{X} \tag{48}$$

Se considera un vector de pornire arbitrar $\mathbf{X}^{(0)}$ si scriem ca:

$$\begin{aligned} \mathbf{AX}^{(0)} &= \lambda^{(1)}\mathbf{X}^{(1)} \\ \mathbf{AX}^{(1)} &= \lambda^{(2)}\mathbf{X}^{(2)} \\ &\ldots \\ \mathbf{AX}^{(m-1)} &= \lambda^{(m)}\mathbf{X}^{(m)} \end{aligned} \tag{49}$$

Procesul iterativ se incheie cind:

$$\left\|\mathbf{X}^{(m)} - \mathbf{X}^{(m-1)}\right\| \le \varepsilon \tag{50}$$

Valorile proprii ale matricei A fiind ordonate:

$$\lambda_1 > \lambda_2 > \ldots > \lambda_n \tag{51}$$

procesul iterativ converge catre λ_1 si vectorul propriu asociat $\mathbf{X}_1$, adica metoda converge catre modul "1" caracterizat de valoarea proprie maxima si vectorul propriu asociat:

$$\lambda^{(m)} = \lambda_1; \;\; \mathbf{X}^{(m)} = \mathbf{X}_1 = \begin{bmatrix} x_{11} & x_{21} & \ldots & x_{n1} \end{bmatrix}^T \tag{52}$$

Pentru demonstratie, ne referim la cazul cind matricea A are valori proprii distincte rezultind ca are vectorii proprii liniar independenti, pot constitui o baza a spatiului vectorial generat de acestia, astfel ca vectorul de pornire $\mathbf{X}^{(0)}$ poate fi considerat o combinatie liniara a acestor vectori proprii:

$$\mathbf{X}^{(0)} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_n\mathbf{X}_n \tag{53}$$

unde c_1, c_2, ..., c_n sunt constante arbitrare.

Reluind opertaille de iterare, se multiplica vectorul $\mathbf{X}^{(0)}$ la stinga, cu matricea **A** scriind:

$$\mathbf{AX}^{(0)} = c_1\mathbf{AX}_1 + c_2\mathbf{AX}_2 + \ldots + c_n\mathbf{AX}_n = c_1\lambda_1\mathbf{X}_1 + c_2\lambda_2\mathbf{X}_2 + \ldots + c_n\lambda_n\mathbf{X}_n = \lambda^{(1)}\mathbf{X}^{(1)} \tag{54}$$

deoarece $\mathbf{AX}_j = \lambda_j\mathbf{X}_j$, $j = 1,2,\ldots,n.$

Vectorul $\mathbf{X}^{(1)}$ obtinut dupa prima operatie de iterare se poate scrie, notind $c_j^{'} = \dfrac{c_j}{\lambda^{(1)}}$:

$$\mathbf{X}^{(1)} = c_1^{'}\lambda_1\mathbf{X}_1 + c_2^{'}\lambda_2\mathbf{X}_2 + \ldots + c_n^{'}\lambda_n\mathbf{X}_n \tag{55}$$

Reluind operatiile de multiplicare cu matricea **A** se obtine:

$$\mathbf{AX}^{(1)} = c_1^{'}\lambda_1\mathbf{AX}_1 + c_2^{'}\lambda_2\mathbf{AX}_2 + \ldots + c_n^{'}\lambda_n\mathbf{AX}_n = c_1^{'}\lambda_1^2\mathbf{X}_1 + c_2^{'}\lambda_2^2\mathbf{X}_2 + \ldots + c_n^{'}\lambda_n^2\mathbf{X}_n = \lambda^{(2)}\mathbf{X}^{(2)}$$

deoarece $\mathbf{AX}_j = \lambda_j\mathbf{X}_j$ $j = 1,2,\ldots,n$. Notind $c_j^{''} = \dfrac{c_j^{'}}{\lambda^{(2)}}$ avem:

$$\mathbf{X}^{(2)} = c_1^{''}\lambda_1^2\mathbf{X}_1 + c_2^{''}\lambda_2^2\mathbf{X}_2 + \ldots + c_n^{''}\lambda_n^2\mathbf{X}_n \tag{56}$$

A *m*-a iteratie se scrie:

$$\mathbf{X}^{(m)} = c_1^{(m)}\lambda_1^m\mathbf{X}_1 + c_2^{(m)}\lambda_2^m\mathbf{X}_2 + \ldots + c_n^{(m)}\lambda_n^m\mathbf{X}_n \tag{57}$$

Cum

$$\lambda_1 > \lambda_2 > \ldots > \lambda_n \tag{58}$$

rezulta ca

$$\lambda_1^m >> \lambda_2^m >> \ldots >> \lambda_n^m \tag{59}$$

ceea ce inseamna ca in exprimarea lui $\mathbf{X}^{(m)}$, termenul dominant este $c_1^m\lambda_1^m\mathbf{X}_1$, adica pentru *m* suficient de mare, procesul iteartiv converge catre valoarea proprie maxima λ_1.